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Quantum corrections to the energy density of a homogeneous Bose gas

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Abstract. Quantum corrections to the properties of a homogeneous interacting Bose gas at zero temperature can be calculated as a low-density expansion in powers of $\sqrt{\rho a^3}$, where ρ is the number density and a is the S-wave scattering length. We calculate the ground state energy density to second order in $\sqrt{\rho a^3}$. The coefficient of the ρa^3 correction has a logarithmic term that was calculated in 1959. We present the first calculation of the constant under the logarithm. The constant depends not only on a, but also on an extra parameter that describes the low energy 3 → 3 scattering of the bosons. In the case of alkali atoms, we argue that the second order quantum correction is dominated by the logarithmic term, where the argument of the logarithm is $\rho a \ell_{\rm V}^2$, and $\ell_{\rm V}$ is the length scale set by the van der Waals potential.

PACS. 03.75.Fi Phase coherent atomic ensemble; quantum condensation phenomena – 05.30.Jp Boson systems

1 Introduction

The successful achievement of Bose-Einstein condensation of atomic gases in magnetic traps [1–3] has created a revival of interest in Bose gases. While a qualitative description of the condensation can be obtained using mean field methods [4], a more quantitative treatment requires including corrections from quantum fluctuations around the mean-field. The relative magnitude of these corrections grows as the square root of the number density of the atoms. They will therefore become more important as higher condensate densities are achieved.

In order to develop a deeper understanding of the quantum fluctuations, it is worthwhile to go back to the simpler problem of a homogeneous gas of interacting bosons at zero temperature. This problem was studied intensively in the 1950's [5,6]. The properties of the system can be calculated as an expansion in powers of $\sqrt{\rho a^3}$, where ρ is the number density of atoms and a is their S-wave scattering length. For example, the expansion for the energy density has the form

$$\mathcal{E}(\rho) = \frac{2\pi\hbar^2 a\rho^2}{m}\left\{1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \left[\frac{8(4\pi-3\sqrt{3})}{3}\ln(\rho a^3) + C\right]\rho a^3 + \dots\right\}. \quad (1)$$

The coefficient of $\sqrt{\rho a^3}$ was first obtained by Lee and Yang for a hard sphere gas [5]. The ρa^3 correction is the first term in the expansion that is sensitive to atomic parameters other than the scattering length. The coefficient of $\ln(\rho)$ in the ρa^3 correction was calculated by Wu, by Hugenholtz and Pines, and by Sawada in 1959 [6].

In this paper, we present the first calculation of the constant C under the logarithm in (1). We express the constant in terms of a coupling constant g_3 that is defined in terms of the low-energy behavior of the amplitude for the 3 → 3 scattering of atoms in the vacuum. The scattering length a and the coupling constant g_3 are the only atomic physics parameters that are needed to calculate the energy density to second order in the quantum corrections. Since g_3 is difficult to measure experimentally or calculate theoretically, it must be treated as a phenomenological parameter. The dependence on this undetermined parameter creates a large uncertainty in the second order quantum correction, except in cases where the correction is dominated by the logarithmic term. We argue that this will typically be the case for alkali atoms, provided we take the argument of the logarithm to be $\rho a \ell_{\rm V}^2$, where $\ell_{\rm V}$ is the length scale set by the van der Waals potential. In this case our result reduces to

$$\mathcal{E}(\rho) = \frac{2\pi\hbar^2 a\rho^2}{m}\left\{1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \frac{8(4\pi-3\sqrt{3})}{3}\ln\left(\rho a \ell_{\rm V}^2\right)\rho a^3 + \dots\right\}. \quad (2)$$

If the logarithm is large compared to 1, this should give an accurate estimate of the second order quantum corrections to the energy density.

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We begin in Section 2 by formulating the problem of calculating the energy density as a quantum field theory problem. In Section 3, we set up a perturbative framework and use it to calculate the energy density to second order in the quantum corrections. In Section 4, we calculate the second order quantum correction to the T-matrix element for the $3 \to 3$ scattering of atoms in the vacuum. We define the coupling constant g_3 and show that the renormalization of g_3 is necessary to remove a logarithmic ultraviolet divergence from the T-matrix element. In Section 5, we complete the calculation of the energy density by including the contribution from g_3 and its renormalization. We then discuss the case of alkali atoms, and argue that the dependence on g_3 can be eliminated in favor of a logarithmic dependence on the length scale set by the van der Waals interaction. The two-loop Feynman diagrams that contribute to the energy density are calculated in the Appendix.

2 Field theory formulation

We are interested in computing the ground state energy density $\mathcal{E}$ of a homogeneous Bose gas as a function of its density ρ. Our starting point is a local quantum field theory that describes atoms with momenta much lower than the inverse of the range of the interatomic potential, which is several Å in the case of alkali atoms. At such low energies, the interactions appear pointlike on the scale of the de Broglie wavelengths of the atoms. The many-body quantum mechanics of the atoms can therefore be formulated in terms of a quantum field theory whose hamiltonian density is a local function of the field:

$$\mathcal{H} = \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + \frac{1}{4}g(\psi^\dagger\psi)^2 + \frac{1}{36}g_3(\psi^\dagger\psi)^3 + \cdots \quad (3)$$

For simplicity, we have assumed that the atoms have only one spin state so that they can be represented by a single complex field $\psi(\mathbf{x}, t)$. The $(\psi^\dagger\psi)^2$ term represents $2 \to 2$ scattering through an S-wave interaction with scattering length a given by

$$g = \frac{8\pi\hbar^2 a}{m} \cdot \quad (4)$$

This coupling constant contains all the information about atomic interactions that is required to calculate the first order quantum corrections to the properties of a sufficiently cold and dilute Bose gas. We follow the effective-field-theory philosophy [7] of including in the hamiltonian all possible local terms that are consistent with the symmetries, which include Galilean invariance and the phase symmetry $\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$. The term $(\psi^\dagger\psi)^3$ in (3) allows $3 \to 3$ scattering through a pointlike interaction. The ...'s in (3) include all possible terms that are higher order in the derivatives or in the number of fields. In principle, the coefficients of these terms can be calculated from the n-body potentials that describe interatomic interactions. In the absence of such calculations, they can be taken as phenomenological parameters. The effective-field-theory philosophy is based on the assumption that there is a systematic expansion in powers of the momentum. The relative importance of the interactions terms in (3) at a low momentum scale p is then given by the dimensionless combination of the coupling constant and p, which is mgp and mg_3p^4 for the $(\psi^\dagger\psi)^2$ term and the $(\psi^\dagger\psi)^3$ term, respectively. If p is sufficiently small, the effects of the g_3 term will be much smaller than those of the g term. Terms with more derivatives or with higher powers of ψ give effects that are suppressed by even more powers of p. Of the infinitely many terms in (3), there are only a finite number that contribute at any given order in p. In the case of the energy density, the appropriate momentum scales are $\rho^{1/3}$ and $(\rho a)^{1/2}$, so the momentum expansion becomes an expansion in powers of the density. We will find that the scattering length g and the coupling constant g_3 are the only atomic physics parameters that contribute to the energy density through third order in ρ [8].

At a given order in p, only a finite number of terms contribute. By tuning the coefficients of these terms, one can describe $n \to n$ scattering of atoms in the vacuum with whatever accuracy is desired.

The phase symmetry $\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$ of the hamiltonian implies the conservation of the number of atoms. The number density operator is $\mathcal{N} = \psi^\dagger\psi$. A homogeneous Bose gas can be described by a field theory with hamiltonian density $\mathcal{H} - \mu\mathcal{N}$, where μ is the chemical potential. The energy density $\mathcal{E}(\mu)$ and the number density $\rho(\mu)$ in the ground state of this field theory are

$$\mathcal{E}(\mu) = \langle\mathcal{H}\rangle_\mu, \quad (5)$$

$$\rho(\mu) = \langle\mathcal{N}\rangle_\mu, \quad (6)$$

where $\langle\cdots\rangle_\mu$ denotes the expectation value in the ground state. By eliminating μ from the two equations (5, 6), we obtain $\mathcal{E}$ as a function of ρ. It is simpler in perturbative calculations to first calculate the free energy density $\mathcal{F}(\mu)$ of the ground state:

$$\mathcal{F}(\mu) \equiv \langle\mathcal{H} - \mu\mathcal{N}\rangle_\mu. \quad (7)$$

After inverting (6) to obtain μ as a function of ρ, one can obtain the energy density from

$$\mathcal{E} = \mathcal{F} + \mu\rho. \quad (8)$$

The partition function for the field theory with hamiltonian density $\mathcal{H} - \mu\mathcal{N}$ can be expressed as a functional integral:

$$\mathcal{Z} = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi \exp\left\{\mathrm{i}S[\psi]\right\}, \quad (9)$$

where the action $S[\psi]$ is given by

$$S[\psi] = \int \mathrm{d}t \int \mathrm{d}^3x \left\{\psi^\dagger\left(\mathrm{i}\partial_t + \frac{\boldsymbol{\nabla}^2}{2m} + \mu\right)\psi - \frac{1}{4}g(\psi^\dagger\psi)^2 - \frac{1}{36}g_3(\psi^\dagger\psi)^3 - \dots\right\}. \quad (10)$$

We have set $\hbar = 1$ in the action. Dimensional analysis can be used to reinsert the factors of $\hbar$ at the end of the calculation. The ...'s in (10) represents all possible terms with higher powers of ψ or more factors of $\boldsymbol{\nabla}$. They also include counterterms that are needed to cancel ultraviolet divergences associated with the parameters μ, g, and g_3. The free energy density $\mathcal{F}(\mu)$ is related to the partition function by

$$\mathcal{Z} = \exp\left\{-\mathrm{i}VT\mathcal{F}(\mu)\right\}, \tag{11}$$

where VT is the spacetime volume. The ground state expectation value of an operator $\langle\mathcal{O}\rangle$ can be expressed as a functional integral:

$$\langle\mathcal{O}\rangle_\mu = \frac{1}{\mathcal{Z}}\int \mathcal{D}\psi^\dagger\mathcal{D}\psi\mathcal{O}\exp\{\mathrm{i}S[\psi]\}. \tag{12}$$

By differentiating the logarithm of both (9, 11) with respect to μ and using (12), we obtain the relation

$$\rho(\mu) = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\mu}(\mu). \tag{13}$$

Differentiating (8) with respect to ρ and using (13), we obtain

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\rho}(\rho) = \mu(\rho). \tag{14}$$

The simplest way to calculate the energy density is to first calculate $\mathcal{F}(\mu)$, use (13) to get $\rho(\mu)$, invert to get $\mu(\rho)$, and then integrate (14) to get $\mathcal{E}(\rho)$.

It is convenient to parameterize the quantum field $\psi(\mathbf{r},t)$ in terms of two real-valued quantum fields ξ and η that describe quantum fluctuations around an arbitrary constant background v:

$$\psi(\mathbf{r},t) = v + \frac{\xi(\mathbf{r},t) + \mathrm{i}\eta(\mathbf{r},t)}{\sqrt{2}}\,. \tag{15}$$

After inserting the field parameterization (15) into the action (10), it can be expanded in powers of the quantum fields ξ and η. By separating the action into a free part and an interaction part, we can express the thermodynamic functions as diagrammatic expansions. The free energy density $\mathcal{F}$ is the sum of all *connected vacuum diagrams*, which are Feynman diagrams with no external legs. This sum is independent of the arbitrary background v. It is convenient to define the thermodynamic potential $\Omega(\mu,v)$, which is the sum of all *one-particle-irreducible* vacuum diagrams. The thermodynamic potential, which depends on v, contains the information required to determine all of the thermodynamic functions. The free energy $\mathcal{F}(\mu)$ can be obtained by evaluating $\Omega(\mu,v)$ at a particular value of v given by the *tadpole condition*

$$\overline{v}(\mu) = \langle\psi\rangle_\mu. \tag{16}$$

For this value of v, those diagrams that can be disconnected by cutting a single line vanish. Thus the sum of connected vacuum diagram reduces to the sum of one-particle-irreducible vacuum diagrams and we have

$$\mathcal{F}(\mu) = \Omega(\mu,\overline{v}(\mu)). \tag{17}$$

Using (15), the tadpole condition (16) reduces to $\langle\xi\rangle_\mu = \langle\eta\rangle_\mu = 0$. The phase of the field ψ can be chosen so that $\langle\eta\rangle_\mu$ is automatically 0. The condition $\langle\xi\rangle_\mu = 0$ can be conveniently expressed in terms of the thermodynamic potential itself:

$$\frac{\partial\Omega}{\partial v}(\mu,\overline{v}(\mu)) = 0. \tag{18}$$

Differentiating both sides of (17) with respect to μ and using (18), we obtain

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\mu}(\mu) = \frac{\partial\Omega}{\partial\mu}(\mu,\overline{v}(\mu)). \tag{19}$$

Comparing with (13), we find that the number density can be expressed as

$$\rho(\mu) = -\frac{\partial\Omega}{\partial\mu}(\mu,\overline{v}(\mu)). \tag{20}$$

3 Ground state energy density

In this section, we calculate the ground state energy for a homogeneous Bose gas to second order in the quantum corrections. We first set up a perturbative framework for carrying out calculations in the presence of a nonzero chemical potential. We use the framework to calculate the energy density to second order in the quantum corrections. We then carry out the renormalizations of μ and g that are necessary to remove power ultraviolet divergences from the energy density.

3.1 Perturbative framework

We can describe a Bose gas with nonzero density ρ by the action (10) with an appropriately chosen value of the chemical potential. For simplicity, we set $g_3 = 0$ and omit all terms in (10) that are higher order in ψ or $\boldsymbol{\nabla}$. We ignore for the moment the counterterms associated with renormalization, so the parameters μ and g should be regarded as bare parameters. Inserting the field parameterization (15) into the action and expanding in powers of ξ and η, the action becomes

$$\begin{aligned} S[\psi] = \int \mathrm{d}t \int \mathrm{d}^3x \biggl\{ & \mu v^2 - \frac{1}{4}gv^4 + \frac{vX}{\sqrt{2}m}\xi \\ & + \frac{1}{2}\left(\eta\dot{\xi} - \xi\dot{\eta}\right) + \frac{1}{4m}\xi\left(\boldsymbol{\nabla}^2 - 2mgv^2 + X\right)\xi \\ & + \frac{1}{4m}\eta\left(\boldsymbol{\nabla}^2 + X\right)\eta \\ & - \frac{gv}{\sqrt{8}}\xi\left(\xi^2+\eta^2\right) - \frac{g}{16}\left(\xi^2+\eta^2\right)^2 \biggr\}, \end{aligned} \tag{21}$$

where $\dot{f} \equiv \frac{\partial}{\partial t} f$ and

$$X = 2m\left(\mu - \frac{1}{2} g v^2\right). \tag{22}$$

To organize the quantum corrections into a loop expansion, we separate the terms in the action (21) that depend on ξ and η into a free part and an interaction part:

$$S[\psi] = S[v] + S_{\mathrm{free}}[\xi,\eta] + S_{\mathrm{int}}[v,\xi,\eta]. \tag{23}$$

The free part of the action consists of the terms quadratic in ξ and η:

$$S_{\mathrm{free}}[\xi,\eta] = \int \mathrm{d}t \int \mathrm{d}^3x \left\{ \frac{1}{2}\left(\eta\dot{\xi} - \xi\dot{\eta}\right) + \frac{1}{4m}\xi(\boldsymbol{\nabla}^2 - 2mgv^2 + X)\xi + \frac{1}{4m}\eta(\boldsymbol{\nabla}^2 + X)\eta \right\}. \tag{24}$$

The Fourier transform of the propagator for the fields ξ and η is a 2×2 matrix:

$$D(\omega,k,v) = \frac{\mathrm{i}}{\omega^2 - \varepsilon^2(\mathrm{k},\mathrm{v}) + \mathrm{i}\epsilon} \times \begin{pmatrix} (k^2 - X)/2m & -\mathrm{i}\omega \\ \mathrm{i}\omega & (k^2 + 2mgv^2 - X)/2m \end{pmatrix}, \tag{25}$$

where $\mathbf{k}$ is the wavevector, ω is the frequency, and

$$\varepsilon^2(k,v) = \frac{1}{4m^2}(k^2 + 2mgv^2 - X)(k^2 - X). \tag{26}$$

The diagonal elements of the propagator matrix (25) are represented by solid lines for ξ and dashed lines for η, as illustrated in Figures 1a and 1b. The off-diagonal elements are represented by a line that is half solid and half dashed, as in Figure 1c. All the remaining terms in the action (21) are treated as interactions:

$$S_{\mathrm{int}}[v,\xi,\eta] = \int \mathrm{d}t \int \mathrm{d}^3x \left\{ \frac{vX}{\sqrt{2m}}\xi - \frac{gv}{\sqrt{8}}\xi\left(\xi^2 + \eta^2\right) - \frac{g}{16}\left(\xi^2 + \eta^2\right)^2 \right\}. \tag{27}$$

The term proportional to ξ is represented by a dot at which a solid line terminates as illustrated in Figure 1d. The 3 and 4-point couplings are represented by points that connect three and four lines, respectively, as in Figures 1e–1i.

It is possible to diagonalize the propagator matrix (25) by applying a Bogoliubov transformation to the fields ξ and η. However, such a transformation makes the interaction terms in the action significantly more complicated and increases the number of diagrams that contribute to most quantities. For explicit calculations, it is more economical to minimize the number of diagrams. We therefore prefer to use a propagator matrix with off-diagonal elements. This perturbative framework was recently used by Haugset, Haugerud, and Ravndal [9] to reproduce the $\sqrt{\rho a^3}$ correction in the expression (1) for the energy density.

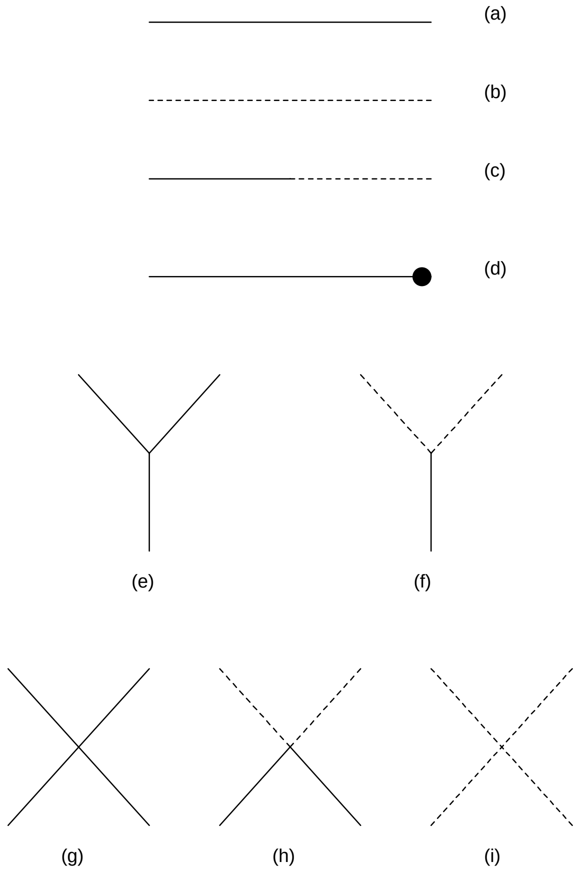


Fig. 1. Propagators and interaction vertices for the real-valued fields ξ and η.

3.2 Free energy density

If the n-loop contribution to the thermodynamic potential Ω is denoted by $\Omega_n(\mu,v)$, the loop expansion for the free energy density (17) is

$$\mathcal{F}(\mu) = \Omega_0(\mu,\overline{v}) + \Omega_1(\mu,\overline{v}) + \Omega_2(\mu,\overline{v}) + \cdots, \tag{28}$$

where $\overline{v}$ is the condensate, which satisfies (18):

$$\frac{\partial\Omega_0}{\partial v}(\mu,\overline{v}) + \frac{\partial\Omega_1}{\partial v}(\mu,\overline{v}) + \cdots = 0. \tag{29}$$

The loop expansion (28) does not coincide with the expansion in the order of quantum corrections because of its dependence on $\overline{v}$. To obtain the quantum expansion, we must expand the condensate $\overline{v}$ around its classical value $\overline{v}_0$, which satisfies

$$\frac{\partial\Omega_0}{\partial v}(\mu,\overline{v}_0) = 0. \tag{30}$$

By expanding (29) in powers of $\overline{v} - \overline{v}_0$, and solving for $\overline{v}$, we obtain the quantum expansion for the condensate:

$$\overline{v}(\mu) = \overline{v}_0(\mu) + \overline{v}_1(\mu) + \overline{v}_2(\mu) + \cdots, \tag{31}$$

where $\overline{v}_n$ is the n-th order quantum correction. For example, the first-order quantum correction is

$$\overline{v}_1(\mu) = -\frac{\partial \Omega_1}{\partial v}(\mu, \overline{v}_0) \Bigg/ \frac{\partial^2 \Omega_0}{\partial v^2}(\mu, \overline{v}_0) \, . \tag{32}$$

Inserting (31) into (28) and expanding in powers of $\overline{v}_1$, $\overline{v}_2$, …, we obtain the quantum expansion for the free energy. Keeping only terms through second order, we have

$$\begin{aligned}\mathcal{F}(\mu) &= \Omega_0(\mu, \overline{v}_0) + \Omega_1(\mu, \overline{v}_0) \\ &\quad + \left(\Omega_2(\mu, \overline{v}_0) + \overline{v}_1 \frac{\partial \Omega_1}{\partial v}(\mu, \overline{v}_0) + \frac{1}{2}\overline{v}_1^2 \frac{\partial^2 \Omega_0}{\partial v^2}(\mu, \overline{v}_0) \right) .\end{aligned} \tag{33}$$

The mean-field contribution to $\Omega(\mu, v)$ is given by the terms in (21) that are independent of ξ and η:

$$\Omega_0(\mu, v) = -\mu v^2 + \frac{1}{4} g v^4 . \tag{34}$$

One solution to (30) is $\overline{v}_0 = 0$, but it is a local maximum of Ω_0 and therefore represents an unstable configuration. The stable solution is

$$\overline{v}_0^2 = 2\mu/g . \tag{35}$$

The tree-level contribution to $\mathcal{F}(\mu)$ given in (33) is

$$\Omega_0(\mu, \overline{v}_0) = -\frac{\mu^2}{g} \cdot \tag{36}$$

The dispersion relation (26) simplifies significantly at the point $v = \overline{v}_0$, because $X = 0$ at that point. It reduces to the Bogoliubov dispersion relation:

$$\varepsilon(k) \equiv \varepsilon(k, \overline{v}_0) = \frac{k\sqrt{k^2 + \Lambda^2}}{2m} , \tag{37}$$

where $\Lambda^2 = 4m\mu$. The tadpole interaction in (27) also vanishes when $v = \overline{v}_0$.

Using the free part of the action (24), we can obtain the one-loop contribution to $\Omega(\mu, v)$:

$$\Omega_1(\mu, v) = \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \int \frac{\mathrm{d}\omega}{2\pi} \ln \det D(\omega, k, v) , \tag{38}$$

where $D(\omega, k, v)$ is given in (25). By integrating over ω, we obtain

$$\Omega_1(\mu, v) = \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \varepsilon(k, v) , \tag{39}$$

where $\varepsilon(k, v)$ is given by (26). The one-loop contribution to the free energy is

$$\Omega_1(\mu, \overline{v}_0) = \frac{1}{4m} I_{0,-1}(4m\mu) , \tag{40}$$

where $I_{0,-1}$ is a function of $\Lambda^2 = 4m\mu$ defined by the integral

$$I_{m,n}(\Lambda^2) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{(p^2)^m}{[2m\varepsilon(p)]^n} . \tag{41}$$

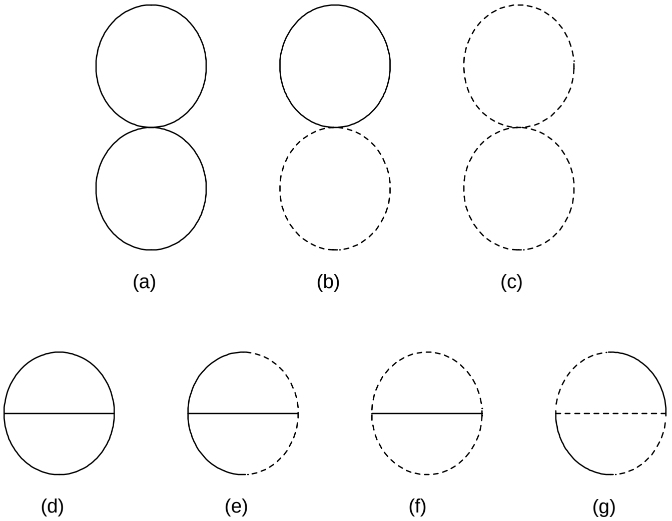


Fig. 2. Two-loop vacuum diagrams that contribute to the thermodynamical potential.

Differentiating the expression (39) with respect to v and evaluating at $v = \overline{v}_0$, we obtain

$$\frac{\partial \Omega_1}{\partial v}(\mu, \overline{v}_0) = \frac{g\overline{v}_0}{4}(3I_{1,1} + I_{-1,-1}) . \tag{42}$$

The first order quantum correction to the condensate, which is given by (32) is

$$\overline{v}_1 = -\frac{g\overline{v}_0}{16\mu}(3I_{1,1} + I_{-1,-1}) . \tag{43}$$

The two-loop contribution to the thermodynamic potential is obtained from the vacuum diagrams shown in Figure 2. The contributions from the individual diagrams are given in Appendix A.5. The sum of the diagrams gives

$$\begin{aligned}\Omega_2(\mu, \overline{v}_0) &= \frac{mg\mu}{8} J \\ &\quad + \frac{g}{64}\left[3I_{-1,-1}^2 + 2I_{-1,-1}I_{1,1} + 3I_{1,1}^2 \right] ,\end{aligned} \tag{44}$$

where

$$J = 6J_{0,0,1} - J_{-1,-1,1} - 3J_{1,1,1} - 2J_{-1,0,0} . \tag{45}$$

The integrals $J_{l,m,n}$ are functions of $\Lambda^2 = 4m\mu$ defined by

$$\begin{aligned}&J_{l,m,n}(\Lambda^2) = \\ &\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\left[p^2/2m\varepsilon(p)\right]^l \left[q^2/2m\varepsilon(q)\right]^m \left[r^2/2m\varepsilon(r)\right]^n}{2m[\varepsilon(p) + \varepsilon(q) + \varepsilon(r)]} ,\end{aligned} \tag{46}$$

where $r = |\mathbf{p} + \mathbf{q}|$.

Inserting (36, 40, 42, 43), and (44) into (33), we obtain the complete expression for the free energy density to second order in the quantum corrections:

$$\mathcal{F}(\mu) = -\frac{\mu^2}{g} + \frac{1}{4m} I_{0,-1}(4m\mu) + \frac{mg\mu}{8} J + \frac{g}{32}\left[I^2_{-1,-1} - 2I_{-1,-1}I_{1,1} - 3I^2_{1,1}\right]. \tag{47}$$

The free energy (47) depends on μ through the explicit factors of μ and through the momentum scale of the integrals which is $\Lambda^2 = 4m\mu$. We have made the argument of the integral explicit for the $I_{0,-1}$ term in (47).

3.3 Energy density

To calculate the energy density, we use (13) to get $\rho(\mu)$, invert that relation to get $\mu(\rho)$, and integrate using (14) to get $\mathcal{E}(\rho)$. Differentiating (47) and using (13), the number density is

$$\rho(\mu) = \frac{2\mu}{g} - \frac{1}{2} I_{1,1}(4m\mu) - \frac{mg}{8}[J + 4m\mu J' + (I_{-1,-1} - I_{1,1})I_{0,1} + (I_{-1,-1} + 3I_{1,1})I_{2,3}], \tag{48}$$

where $J' = \mathrm{d}J/\mathrm{d}\Lambda^2$ and all the integrals are functions of $\Lambda^2 = 4m\mu$. We have made the argument of the integral explicit for the $I_{1,1}$ term. We have used the identity (A.6) to differentiate the integrals $I_{m,n}$. Inverting (48) to get μ as a function of ρ, we obtain

$$\mu(\rho) = \frac{1}{2} g\rho + \frac{1}{4} g I_{1,1}(2mg\rho) + \frac{mg^2}{16}[J + 2mg\rho J' + (I_{-1,-1} - I_{1,1})I_{0,1} + (I_{-1,-1} + I_{1,1})I_{2,3}], \tag{49}$$

where the integrals are now functions of $\Lambda^2 = 2mg\rho$. Using the identity (A.6), we can write the expression as a total derivative:

$$\mu(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\frac{1}{4} g\rho^2 + \frac{1}{4m} I_{0,-1}(2mg\rho) + \frac{mg^2\rho}{16} J + \frac{g}{32}[I^2_{-1,-1} - 2I_{-1,-1}I_{1,1} - I^2_{1,1}]\right\}. \tag{50}$$

We can now read off the energy density using (14):

$$\mathcal{E}(\rho) = \mathcal{E}_0 + \frac{1}{4} g\rho^2 + \frac{1}{4m} I_{0,-1}(2mg\rho) + \frac{mg^2\rho}{16} J + \frac{g}{32}[I^2_{-1,-1} - 2I_{-1,-1}I_{1,1} - I^2_{1,1}], \tag{51}$$

where $\mathcal{E}_0$ is an integration constant and all the integrals are functions of $\Lambda^2 = 2mg\rho$. It is convenient to choose the integration constant $\mathcal{E}_0$ so that the energy of the vacuum is zero: $\mathcal{E}(0) = 0$.

3.4 Renormalization of μ and g

Our result (51) for the energy density can be generalized to an arbitrary number of spatial dimensions D simply by replacing the integration measure $\mathrm{d}^3p/(2\pi)^3$ in (41, 46) by $\mathrm{d}^Dp/(2\pi)^D$. The integrals $I_{m,n}$ and $J_{l,m,n}$ in (51) are ultraviolet divergent for any positive number of dimensions D. If we impose a momentum cutoff Λ_{UV}, then $I_{0,-1}$ diverges like $\Lambda_{\mathrm{UV}}^{D+2}$, while $I_{-1,-1}$ and $I_{1,1}$ diverge like $\Lambda_{\mathrm{UV}}^{D}$. The integrals $J_{l,m,n}$ contain subintegrals that diverge like $\Lambda_{\mathrm{UV}}^{D-2}$, and, if $D > 1$, they also have an overall divergence that scales like $\Lambda_{\mathrm{UV}}^{2D-2}$. There are cancellations among the $J_{l,m,n}$ integrals that reduce the overall divergence to $\Lambda_{\mathrm{UV}}^{2D-6}$ for $D > 3$ and $\ln \Lambda_{\mathrm{UV}}$ for $D = 3$.

The divergences can be removed by renormalization. A convenient way to implement the renormalization is to add counterterms to the action (10):

$$\delta S = \int \mathrm{d}t \int \mathrm{d}^3x \left\{\delta\mu\psi^\dagger\psi - \frac{1}{4}\delta g(\psi^\dagger\psi)^2 + \cdots\right\}. \tag{52}$$

In perturbative calculations, the counterterms $\delta\mu$ and δg should be treated as quantum corrections. They can be expanded according to the order in the quantum correction:

$$\delta\mu = \delta_1\mu + \delta_2\mu + \cdots, \tag{53}$$

$$\delta g = \delta_1 g + \delta_2 g + \cdots. \tag{54}$$

To obtain the free energy after the renormalizations of μ and g, we substitute $\mu \to \mu + \delta\mu$ and $g \to g + \delta g$ into (47) and expand in the order of the quantum correction. The complete expression to second order in the quantum corrections is

$$\begin{aligned}\mathcal{F}(\mu) = -\frac{\mu^2}{g} &+ \left[\frac{1}{4m} I_{0,-1}(4m\mu) - 2\frac{\mu}{g}\delta_1\mu + \frac{\mu^2}{g^2}\delta_1 g\right] \\ &+ \left[\frac{mg\mu}{8} J + \frac{g}{32}(I^2_{-1,-1} - 2I_{-1,-1}I_{1,1} - 3I^2_{1,1}) \right. \\ &+ \frac{1}{2} I_{1,1}\delta_1\mu - 2\frac{\mu}{g}\delta_2\mu + \frac{\mu^2}{g^2}\delta_2 g - \frac{1}{g}(\delta_1\mu)^2 \\ &\left. + 2\frac{\mu}{g^2}\delta_1\mu\delta_1 g - \frac{\mu^2}{g^3}(\delta_1 g)^2\right]. \end{aligned} \tag{55}$$

By repeating each of the steps in Section 3.3 including the effects of the counterterms, we obtain an expression for the energy density that takes into account the renormalization of μ and g:

$$\begin{aligned}\mathcal{E}(\rho) = \mathcal{E}_0 + \frac{1}{4} g\rho^2 &+ \left[\frac{1}{4m} I_{0,-1}(2mg\rho) - \rho\delta_1\mu + \frac{\rho^2}{4}\delta_1 g\right] \\ &+ \left[\frac{mg^2\rho}{16} J + \frac{g}{32}(I^2_{-1,-1} - 2I_{-1,-1}I_{1,1} - I^2_{1,1}) \right. \\ &\left. + \frac{\rho}{4} I_{1,1}\delta_1 g - \rho\delta_2\mu + \frac{\rho^2}{4}\delta_2 g\right]. \end{aligned} \tag{56}$$

The ultraviolet divergences in (56) that are independent of ρ can be cancelled by $\mathcal{E}_0$. The counterterms $\delta_1\mu$, $\delta_2\mu$,

$\delta_1 g$, and $\delta_2 g$ can be determined by demanding that the ρ-dependent power ultraviolet divergences cancel. For example, the integral appearing in the first order quantum correction can be written

$$I_{0,-1} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(p\sqrt{p^2 + 2mg\rho} - p^2 - mg\rho + \frac{m^2 g^2 \rho^2}{2p^2} \right) + \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(p^2 + mg\rho - \frac{m^2 g^2 \rho^2}{2p^2} \right) . \tag{57}$$

The first integral converges and each term in the second integral gives a power ultraviolet divergence. The integral of the p^2 term in the second integral of (57) is independent of ρ and can be cancelled by $\mathcal{E}(0)$. The remaining divergences can be cancelled in (56) by taking

$$\delta_1 \mu = \frac{g}{4} \int \frac{\mathrm{d}^D p}{(2\pi)^D}, \tag{58}$$

$$\delta_1 g = \frac{g^2 m}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2}. \tag{59}$$

For later convenience, we have generalized the integrals to an arbitrary number of spatial dimensions D. Similarly we can determine the counterterms $\delta_2 \mu$ and $\delta_2 g$ by demanding the cancellation of ρ-dependent power ultraviolet divergences in the second-order quantum corrections in (56):

$$\delta_2 \mu = \frac{mg^2}{4} \left(\int \frac{\mathrm{d}^D p}{(2\pi)^D} \right) \left(\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} \right), \tag{60}$$

$$\delta_2 g = \frac{m^2 g^3}{8} \left(\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{p^2} \right)^2 . \tag{61}$$

In $D = 3$ dimensions, all the power ultraviolet divergences can be removed by renormalizations of μ, g, and $\mathcal{E}_0$. However there is still a logarithmic ultraviolet divergence coming from the J term in (56). That divergence is the only obstacle to completing our calculation of the energy density to second order in the quantum corrections. The predictive power of quantum field theory lies in the fact that the same renormalizations must remove the ultraviolet divergences from all physical quantities. The renormalizations that remove the ultraviolet divergences from the ground state energy density must also remove ultraviolet divergences from the amplitudes for the low-energy scattering of atoms in the vacuum. In the next section, we calculate quantum corrections to the amplitudes for scattering of atoms in the vacuum. After identifying the renormalization that removes the logarithmic divergence from the energy density, we will complete the calculation of $\mathcal{E}$ in Section 5.

4 Scattering of atoms in the vacuum

In this section, we calculate quantum corrections to the T-matrix elements for 2 → 2 scattering and for 3 → 3 scattering of atoms in the vacuum. We determine the renormalizations that are necessary to remove ultraviolet divergences from these T-matrix elements.

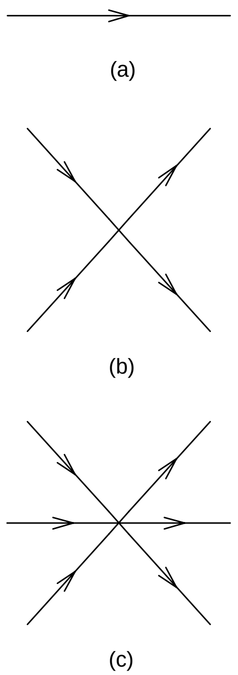


Fig. 3. Propagator and interaction vertices for the complex-valued field ψ.

4.1 Perturbative framework

Atoms in the vacuum can be described by the action (10) with the chemical potential μ set to 0. A perturbative framework for calculating their scattering amplitudes can be obtained by separating the action into a free part and an interaction part as follows:

$$S_{\mathrm{free}}[\psi] = \int \mathrm{d}t \int \mathrm{d}^3 x \psi^\dagger \left(\mathrm{i}\partial_t + \frac{\boldsymbol{\nabla}^2}{2m} \right) \psi, \tag{62}$$

$$S_{\mathrm{int}}[\psi] = \int \mathrm{d}t \int \mathrm{d}^3 x \left(-\frac{1}{4} g (\psi^\dagger \psi)^2 - \frac{1}{36} g_3 (\psi^\dagger \psi)^3 - \ldots \right) . \tag{63}$$

We can read off the Feynman propagator from S_{free}. Its Fourier transform is

$$D(\omega, k) = \frac{\mathrm{i}}{\omega - k^2/2m + \mathrm{i}\epsilon} \cdot \tag{64}$$

This propagator is represented by a solid line with an arrow as illustrated in Figure 3a. The 4-particle and 6-particle interactions in (63) are represented by vertices connecting 4 lines and 6 lines, respectively, as shown in Figure 3b and 3c. These vertices have equal number of arrows entering and exiting. This reflects the conservation of the number of atoms, which follows from the phase symmetry $\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$ of the action consisting of (62, 63).

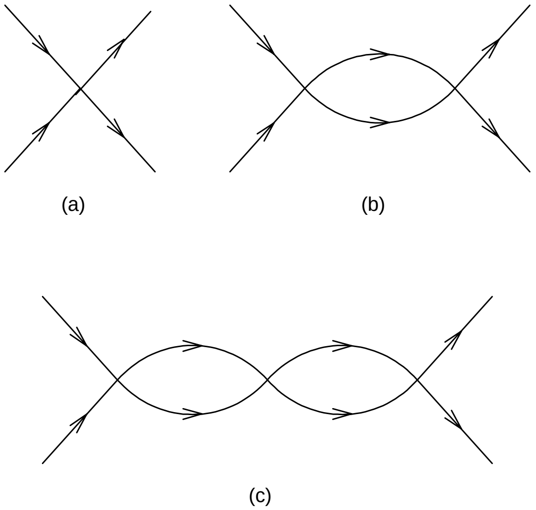


Fig. 4. Diagrams for 2 → 2 scattering: the tree-level diagram (a), a one-loop diagram (b), and a two-loop diagram (c).

4.2 2 → 2 scattering

Two atoms with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ can scatter into states with momenta $\mathbf{k}'_1$ and $\mathbf{k}'_2$ that are allowed by conservation of energy and momentum. The probability amplitude for the scattering process is given by the T-matrix element $\mathcal{T}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}'_1,\mathbf{k}'_2)$.

The only terms in the action that contribute to 2 → 2 scattering are those that are fourth order in ψ. If the only such term is $(\psi^\dagger\psi)^2$, the scattering is purely S-wave. In this case, the T-matrix element is a function of a single variable:

$$\mathcal{T}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}'_1,\mathbf{k}'_2) = \mathcal{T}(q_{12}), \tag{65}$$

where $q_{12} = |\mathbf{k}_1 - \mathbf{k}_2|$. The center-of-mass energy E is related to q_{12} by $E = q_{12}^2/4m$.

The T-matrix element for 2 → 2 scattering can be expanded in the order of quantum corrections:

$$\mathcal{T}(q_{12}) = -g + \mathcal{T}_1(q_{12}) + \mathcal{T}_2(q_{12}) + \ldots. \tag{66}$$

The first term comes from the tree diagram in Figure 4a. The first quantum correction comes from the one-loop diagram in Figure 4b. Using contour integration to evaluate the energy integral, we get

$$\mathcal{T}_1(q_{12}) = \frac{mg^2}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2 - \mathbf{p}\cdot(\mathbf{k}_1+\mathbf{k}_2) + \mathbf{k}_1\cdot\mathbf{k}_2 - \mathrm{i}\epsilon}\cdot \tag{67}$$

The integral has a linear ultraviolet divergence, but the divergence can be removed by renormalization of the coupling constant g. Including the counterterms from the tree diagram in Figure 4a, we get

$$\mathcal{T}_1(q_{12}) = \frac{mg^2}{2} \times\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2 - \mathbf{p}\cdot(\mathbf{k}_1+\mathbf{k}_2) + \mathbf{k}_1\cdot\mathbf{k}_2 - \mathrm{i}\epsilon} - \delta_1 g. \tag{68}$$

To maintain rotational symmetry, we must shift the integration variable by $\mathbf{p} \to \mathbf{p} + (\mathbf{k}_1+\mathbf{k}_2)/2$ before imposing an ultraviolet cutoff $|\mathbf{p}| < \Lambda_{\mathrm{UV}}$. The resulting integral can be evaluated analytically and we obtain

$$\mathcal{T}_1(q_{12}) = \frac{mg^2}{4\pi^2}\Lambda_{\mathrm{UV}} - \frac{mg^2}{16\pi}(-q_{12}^2 - \mathrm{i}\epsilon)^{1/2} - \delta_1 g. \tag{69}$$

The counterterm $\delta_1 g$, whose value was determined in (59), precisely cancels the linear ultraviolet divergence in (69). The final result is

$$\mathcal{T}_1(q_{12}) = \mathrm{i}\frac{mg^2}{16\pi}q_{12}. \tag{70}$$

A particularly convenient method for regularizing ultraviolet divergent integrals is *dimensional regularization*. The number of spatial dimensions D is taken to be a complex variable. The integral is evaluated as a function of D in a region of the complex D-plane where it converges. This defines an analytic function of D which can be analytically continued to $D = 3$. After shifting the integration variable in (67) by $\mathbf{p} \to \mathbf{p} + (\mathbf{k}_1+\mathbf{k}_2)/2$ and then integrating over the angles in D dimensions, we obtain

$$\mathcal{T}_1(q_{12}) = \frac{mg^2}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)}\int_0^\infty \mathrm{d}p\frac{p^{D-1}}{p^2 - q_{12}^2/4 - \mathrm{i}\epsilon}\cdot \tag{71}$$

The integral converges for Re$D < 2$ and is given by

$$\mathcal{T}_1(q_{12}) = \frac{mg^2}{2}\frac{\Gamma(1-D/2)}{(4\pi)^{D/2}}\left(\frac{-q_{12}^2}{4} - \mathrm{i}\epsilon\right)^{(D-2)/2}. \tag{72}$$

Analytically continuing to $D = 3$, we recover the result (70).

One of the great advantages of dimensional regularization is that integrals that contain no scale are set identically equal to 0:

$$\int\frac{\mathrm{d}^Dp}{(2\pi)^D}p^\alpha = 0. \tag{73}$$

This formula can be derived by first integrating over angles in D dimensions, and then separating the integral over p into two pieces corresponding to $p < p^*$ and $p > p^*$. The integral over $p < p^*$ can be evaluated for D large enough that it is convergent in the infrared. The integral over $p > p^*$ can be evaluated for D small enough that it is convergent in the ultraviolet. Upon analytically continuing the two integrals to $D = 3$, we find that they cancel exactly. Because of the identity (73), dimensional regularization sets pure power ultraviolet divergences to 0. Thus the counterterms $\delta\mu$ and δg, which are given by the integrals in (58–61), vanish. With dimensional regularization, the only ultraviolet divergences that require explicit renormalization are logarithmic divergences, which appear as poles in $D - 3$. Nontrivial counterterms are therefore needed only to cancel logarithmic ultraviolet divergences.

The quantum corrections to the scattering amplitude from higher order diagrams, like the two-loop diagram in

Figure 4c, form a geometric series and can be summed up exactly. The nth term in the series is

$$\mathcal{T}_n(q_{12}) = -g\left(-\mathrm{i}\frac{mg}{16\pi}q_{12}\right)^n . \tag{74}$$

Summing up the geometric series, the complete 2 → 2 scattering amplitude is

$$\mathcal{T}(q_{12}) = \frac{-g}{1+\mathrm{i}mgq_{12}/(16\pi)} . \tag{75}$$

The imaginary part of this T-matrix element is precisely that required by the optical theorem.

4.3 3 → 3 scattering

Three atoms with momenta $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ can scatter into states with momenta $\mathbf{k}_1'$, $\mathbf{k}_2'$ and $\mathbf{k}_3'$ that are allowed by conservation of energy and momentum. The probability amplitude for 3 → 3 scattering processes in which all 3 atoms participate is given by the connected T-matrix element, which we denote by $\mathcal{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')$. For simplicity, we consider only the center of momentum frame, where $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$ and we use the shorthand

$$\mathcal{T}(123\to 1'2'3') \equiv \mathcal{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'). \tag{76}$$

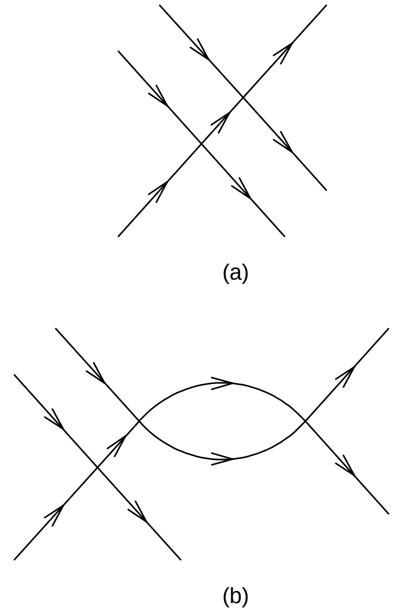
The connected T-matrix element for 3 → 3 scattering can be separated into the terms that involve a single virtual particle in the intermediate state and the remainder, which is called the *one-particle-irreducible* (1PI) part of $\mathcal{T}$:

$$\mathcal{T}(123\to 1'2'3') = \mathcal{T}^{1\mathrm{PI}}(123\to 1'2'3') + \sum_{(123)}\sum_{(1'2'3')}\mathcal{T}(q_{12}) \times \frac{m}{\mathbf{k}_1\cdot\mathbf{k}_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_3'+k_3'^2-\mathrm{i}\epsilon}\mathcal{T}(q_{1'2'}), \tag{77}$$

where $q_{12}=|\mathbf{k}_1-\mathbf{k}_2|$ and $q_{1'2'}=|\mathbf{k}_1'-\mathbf{k}_2'|$. The sums are over cyclic permutations of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ and of $\mathbf{k}_1'$, $\mathbf{k}_2'$, and $\mathbf{k}_3'$. The term in the sum that is given explicitly corresponds to the 2 → 2 scattering of particles 1 and 2 to produce particle 3′ and a virtual particle. A subsequent 2 → 2 scattering of the virtual particle and particle 3 produces particles 1′ and 2′. Examples of diagrams that contribute to the sum are the tree diagram in Figure 5a and the one-loop diagram in Figure 5b.

The leading contributions to the 1PI T-matrix element for 3 → 3 scattering come from one-loop diagrams like the one in Figure 6a. After using contour integration to integrate over the loop energy, we obtain

$$\mathcal{T}_1^{1\mathrm{PI}}(123\to 1'2'3') = -m^2g^3\sum_{(123)}\sum_{(1'2'3')}\mathcal{I}(123\to 1'2'3'), \tag{78}$$


(a)

(b)

Fig. 5. One-particle-reducible diagrams for 3 → 3 scattering: a tree-level diagram (a) and a one-loop diagram (b).

where

$$\mathcal{I}(123\to 1'2'3') = \int\frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{(p^2+\mathbf{p}\cdot\mathbf{k}_3+\mathbf{k}_1\cdot\mathbf{k}_2-i\epsilon)(p^2+\mathbf{p}\cdot\mathbf{k}_3'+\mathbf{k}_1'\cdot\mathbf{k}_2'-i\epsilon)} . \tag{79}$$

This integral is ultraviolet convergent.

The next most important quantum corrections to the 1PI T-matrix element come from two-loop diagrams, such as those in Figure 6b, c, and d, and from the insertion of a counterterm $\delta_1 g$ into the one-loop diagram in Figure 6a. After using contour integration to integrate over the loop energies, we obtain

$$\begin{aligned}\mathcal{T}_2^{1\mathrm{PI}}(123\to 1'2'3') &= \frac{\delta_1 g}{g}\mathcal{T}_1^{1\mathrm{PI}}(123\to 1'2'3') \\ &+m^2g^2\sum_{(123)}\sum_{(1'2'3')}\left[\mathcal{T}_1(q_{12})+\mathcal{T}_1(q_{1'2'})\right]\mathcal{I}(123\to 1'2'3') \\ &+m^3g^4\sum_{(123)}\sum_{(1'2'3')}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{p^2+q^2+r^2-2mE-\mathrm{i}\epsilon} \\ &\times\left\{\frac{2}{(p^2+\mathbf{p}\cdot\mathbf{k}_3+\mathbf{k}_1\cdot\mathbf{k}_2-\mathrm{i}\epsilon)(q^2+\mathbf{q}\cdot\mathbf{k}_3'+\mathbf{k}_1'\cdot\mathbf{k}_2'-\mathrm{i}\epsilon)}\right. \\ &\left.+\frac{1}{(p^2+\mathbf{p}\cdot\mathbf{k}_3+\mathbf{k}_1\cdot\mathbf{k}_2-\mathrm{i}\epsilon)(p^2+\mathbf{p}\cdot\mathbf{k}_3'+\mathbf{k}_1'\cdot\mathbf{k}_2'-\mathrm{i}\epsilon)}\right\},\end{aligned} \tag{80}$$

where $r=|\mathbf{p}+\mathbf{q}|$ and $E=(k_1^2+k_2^2+k_3^2)/2m$ is the total energy. The integral over $\mathbf{q}$ in the last term of (80) has a linear ultraviolet divergence. Using the expression (59)

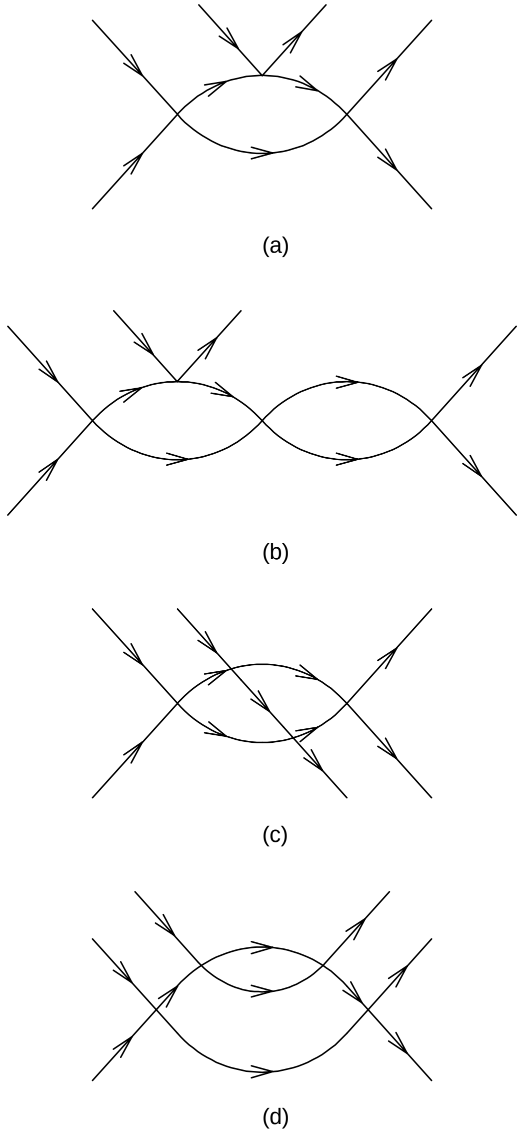


Fig. 6. One-particle-irreducible (1PI) diagrams for 3 → 3 scattering: a one-loop diagram (a) and two-loop diagrams (b), (c), and (d).

for the counterterm $\delta_1 g$, we can see that the first term on the right side of (80) cancels the linear divergence from the integral over $\mathbf{q}$ in the last term.

After cancellation of the linear divergence, there remains an overall logarithmic divergence in the integral over $\mathbf{p}$ and $\mathbf{q}$ in (80). This is evident from scaling $\mathbf{p} \to t\mathbf{p}$ and $\mathbf{q} \to t\mathbf{q}$. As $t \to \infty$, the integrand scales like $1/t^6$ while the integration volume scales like t^6. The divergence is independent of the external momenta and therefore corresponds to a point interaction between the three particles. The divergence can be cancelled by the counterterm δg_3 associated with the $(\psi^\dagger\psi)^3$ term in the action (10). However, if we include that counterterm, we must also for consistency include the contribution to $\mathcal{T}(123 \to 1'2'3')$ from the coupling constant g_3. Thus we must add to (80) the contribution from the tree diagram in Figure 3c:

$$\Delta\mathcal{T} = -(g_3 + \delta g_3). \tag{81}$$

We choose to use dimensional regularization to regularize the integral in (80). The logarithmic ultraviolet divergence then appears as a pole in $D-3$. The pole in the integral over $\mathbf{p}$ and $\mathbf{q}$ in (80) is identical to that of the following integral, which is evaluated in the limit $D \to 3$ in Appendix A.4:

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \int \frac{\mathrm{d}^D q}{(2\pi)^D} \left\{ \frac{2}{(p^2+q^2+r^2+2\kappa^2)(p^2+\kappa^2)(q^2+\kappa^2)} \right.$$
$$\left. + \frac{1}{(p^2+q^2+r^2+2\kappa^2)(p^2+\kappa^2)^2} - \frac{1}{2(q^2+\kappa^2)(p^2+\kappa^2)^2} \right\}$$
$$= -\frac{4\pi - 3\sqrt{3}}{192\pi^3} \left(\frac{1}{D-3} - 1.13459 \right) \kappa^{2(D-3)}. \tag{82}$$

The last term in the integrand of (82) cancels the linear divergence in the integral over $\mathbf{q}$ of the previous term, but does not change the logarithmic ultraviolet divergence, which gives the pole in $D-3$. The pole term in (80) is therefore

$$\left[\mathcal{T}_2^{\mathrm{1PI}}(123 \to 1'2'3')\right]_{\mathrm{pole}} = -\frac{3(4\pi - 3\sqrt{3})}{64\pi^3(D-3)} m^3 g^4. \tag{83}$$

The pole must be cancelled by the counterterm δg_3 in (81). One of the simplest renormalization schemes is *minimal subtraction* [10]. This scheme defines a *running coupling constant* $g_3(\kappa)$ that depends on an arbitrary *renormalization scale* κ. The minimal subtraction prescription is to choose the counterterm to be a pure pole in $D-3$ multiplied by a power of κ:

$$\delta g_3(\kappa) = -\frac{3(4\pi - 3\sqrt{3})}{64\pi^3(D-3)} m^3 g^4 \kappa^{2(D-3)}. \tag{84}$$

The exponent of κ is chosen so that both sides of (84) have the same engineering dimensions even when $D \neq 3$. Without such a factor, renormalized quantities would involve logarithms of dimensionful quantities. The power of κ in (84) is determined by dimensional analysis. With $\hbar$ set equal to 1, the terms in the action must be dimensionless. Time has dimensions mL^2 when $\hbar = 1$, where L refers to length. The integration measure $\int \mathrm{d}t \int \mathrm{d}^D x$ therefore has dimensions mL^{D+2}. Since $\psi^\dagger\psi$ is a number density, ψ has dimensions $[\psi] = L^{-D/2}$. The dimensions of the coupling constants are then $[g] = L^{D-2}/m$ and $[g_3] = L^{2D-2}/m$. The power of κ in (84) provides the extra factor of L^{-2D+6} required for dimensional consistency.

Physical quantities cannot depend on the arbitrary parameter κ introduced through the counterterm (84). The coupling constant $g_3(\kappa)$ must therefore depend on κ in such a way that the combination $g_3 + \delta g_3$ is independent of κ. This statement can be conveniently expressed in the form of a *renormalization group equation*. Using (84), the condition $(\mathrm{d}/\mathrm{d}\kappa)(g_3 + \delta g_3) = 0$ reduces in the limit $D \to 3$ to

$$\kappa \frac{\mathrm{d}}{\mathrm{d}\kappa} g_3 = \frac{3(4\pi - 3\sqrt{3})}{32\pi^3} m^3 g^4. \tag{85}$$

Since m and g are independent of κ, the solution to the equation (85) is

$$g_3(\kappa') = g_3(\kappa) + \frac{3(4\pi - 3\sqrt{3})}{32\pi^3} m^3 g^4 \ln \frac{\kappa'}{\kappa} \cdot \tag{86}$$

This equation tells us that the parameter g_3 is a running coupling constant that varies logarithmically with the renormalization scale κ. The renormalization scale κ can be interpreted as the inverse of the spatial resolution. As κ increases, the spatial resolution becomes finer and part of the "pointlike" 3 → 3 scattering amplitude represented by the diagram in Figure 3c is resolved into the successive 2 → 2 scatterings represented by the diagrams in Figures 6c and 6d. The contributions from the two individual two-loop diagrams have opposite signs and the net effect is that the coupling constant g_3 increases as κ increases.

The renormalized expression for $\mathcal{T}_2^{1\mathrm{PI}}(123 \to 1'2'3')$ is given by the limit as $D \to 3$ of the sum of (80) and (81). To take the limit, we must extract the pole in $D-3$ from the integral over $\mathbf{p}$ and $\mathbf{q}$ in (80), so that it can be cancelled by the counterterm δg_3. This can be accomplished by subtracting the integrand on the left side of (82) from the last integrand in (80), and then adding to (80) the right side of (82) multiplied by $9m^3g^4$. The integral over $\mathbf{p}$ and $\mathbf{q}$ is then convergent in $D = 3$, but depends logarithmically on the scale κ.

We have eliminated the logarithmic ultraviolet divergence from $\mathcal{T}^{1\mathrm{PI}}$ by expressing it in terms of the renormalized coupling constant $g_3(\kappa)$ defined by dimensional regularization and minimal subtraction. The resulting expression for $\mathcal{T}^{1\mathrm{PI}}$ can serve as a definition of $g_3(\kappa)$ that makes no reference to the regularization scheme. Someone who prefers a more physical definition of the coupling constant can define $g_{3,\mathrm{phys}}$ to be equal to the value of $-\mathcal{T}^{1\mathrm{PI}}(123 \to 1'2'3')$ at their favorite configuration of the initial and final momenta. Any such coupling constant can be expressed in the form

$$g_{3,\mathrm{phys}} = g_3(\kappa) + C(\kappa) m^3 g^4. \tag{87}$$

Since its definition makes no reference to the renormalization scale κ, the physical coupling constant satisfies $(\mathrm{d}/\mathrm{d}\kappa) g_{3,\mathrm{phys}} = 0$. The renormalization group equation (85) then implies that the coefficient $C(\kappa)$ in (87) is a linear function of $\ln \kappa$. It therefore vanishes for some value κ_{phys} and we have

$$g_{3,\mathrm{phys}} = g_3(\kappa_{\mathrm{phys}}). \tag{88}$$

Thus any physical definition of the coupling constant is equivalent to the running coupling constant $g_3(\kappa)$ evaluated at a particular value of the renormalization scale κ. Thus there is little to be gained by using a more physical definition of the coupling constant.

The integrals in $\mathcal{T}_2^{1\mathrm{PI}} + \Delta\mathcal{T}$ are functions of the renormalization scale κ and the initial and final momenta. For momentum configurations in which the squares of the momenta and their inner products are all comparable in magnitude, the only scales in the integrands are κ^2 and mE, where E is the total center-of-mass energy. Since the integral varies logarithmically with κ, it must also depend logarithmically on mE. If the ratio of these two scales is sufficiently large, the integral is dominated by the logarithm. However, since the dependence of the logarithm on κ is cancelled by $g_3(\kappa)$, there must be a large cancelling contribution from the g_3 term. The dominant terms in the second order quantum correction to the T-matrix element are therefore

$$\mathcal{T}_2^{1\mathrm{PI}} + \Delta\mathcal{T} \approx -g_3(\kappa) - \frac{3(4\pi - 3\sqrt{3})}{64\pi^3} m^3 g^4 \ln \frac{mE}{\kappa^2}. \tag{89}$$

The large logarithm can be avoided by choosing the renormalization scale of the running coupling constant $g_3(\kappa)$ to be $\kappa = \sqrt{mE}$. Thus the most appropriate choice for the renormalization scale κ in the 3 → 3 scattering amplitude is the magnitude of the typical momenta of the scattering particles. This choice will avoid a large cancellation between the two terms in (89).

5 Renormalized energy density

In this Section, we complete the calculation of the energy density in Section 3 by using the renormalization of g_3 to remove the logarithmic ultraviolet divergences from the second-order quantum correction. Our final result is expressed in terms of parameters g and $g_3(\kappa)$ that can be defined purely in terms of the scattering of atoms. We then discuss the case of alkali atoms, and argue that in this case the dependence on g_3 can be eliminated in favor of a logarithmic dependence on the length scale set by the van der Waals potential.

5.1 First-order quantum correction

To simplify the calculation of the quantum corrections, we use dimensional regularization to regularize ultraviolet divergences and minimal subtraction to carry out the renormalization of the parameter g_3. One of the great advantages of dimensional regularization is that it sets power ultraviolet divergence to 0. The only ultraviolet divergences that require explicit renormalization are logarithmic divergences, which appear as poles in $D-3$. Since the ultraviolet divergences removed by the renormalization of μ and g are power divergences, the explicit renormalization of these parameters is unnecessary with dimensional regularization. This can be seen from the expressions (58–61) for the counterterms, which all vanish by the identity (73). Thus the expression (56) for the energy density after renormalization of μ and g collapses to (51). The integration constant $\mathcal{E}_0$, which is used to set $\mathcal{E}(0) = 0$, is also zero in dimensional regularization. This follows from the fact that the only scale in the dimensionally regularized integrals $I_{m,n}$ and J is $2mg\rho$. These integrals therefore vanish when $\rho = 0$, since there is no momentum scale.

From (51), the first order quantum correction to the energy density is

$$\mathcal{E}_1(\rho) = \frac{1}{4m} I_{0,-1}(2mg\rho). \tag{90}$$

Using the expression for the dimensionally regularized integral given in (A.12), this becomes

$$\mathcal{E}_1(\rho) = \frac{1}{60\pi^2}\frac{(2mg\rho)^{5/2}}{m} \cdot \tag{91}$$

Using (4) to express g in terms of the scattering length a, we recover the classic result for the first order quantum correction given in (1).

5.2 Second-order quantum correction

Dimensional regularization eliminates the power ultraviolet divergences from the second-order quantum correction in (51), but the expression still contains a logarithmic ultraviolet divergence. The renormalization of this divergence requires including the effects of the coupling constant g_3 and its counterterm at tree level. The term that must be added to the free energy density can be read off from the integrand in the action (10):

$$\Delta\mathcal{F}(\mu) = \frac{1}{36}(g_3 + \delta g_3)\overline{v}_0{}^6. \tag{92}$$

Following the effect of the correction through to the energy density, we find

$$\Delta\mathcal{E}(\rho) = \frac{1}{36}(g_3 + \delta g_3)\rho^3. \tag{93}$$

With dimensional regularization, the complete second-order quantum correction is the sum of (93) and the correction in (51):

$$\begin{aligned}\mathcal{E}_2(\rho) = &\frac{1}{36}\left[g_3(\kappa) + \delta g_3(\kappa)\right]\rho^3 \\ &+ \frac{mg^2\rho}{16}J + \frac{g}{32}\left(I^2_{-1,-1} - 2I_{-1,-1}I_{1,1} - I^2_{1,1}\right).\end{aligned} \tag{94}$$

The values of the integrals J, $I_{-1,-1}$, and $I_{1,1}$ in dimensional regularization are given in (A.23), (A.13), and (A.14) with $\Lambda^2 = 2mg\rho$. The pole in $D-3$ in J is cancelled by the pole in the counterterm $\delta g_3(\kappa)$ given in (84), but the cancellation leaves a logarithm of $2mg\rho/\kappa^2$ in the limit $D \to 3$. Combining all the terms, our expression for $\mathcal{E}_2$ is

$$\begin{aligned}\mathcal{E}_2(\rho) = &\frac{1}{36}g_3(\kappa)\rho^3 \\ &+ \frac{4\pi - 3\sqrt{3}}{768\pi^3}\left(\ln\frac{2mg\rho}{\kappa^2} + 0.80\right)m^3g^4\rho^3,\end{aligned} \tag{95}$$

where $g_3(\kappa)$ is the running coupling constant defined by the minimal subtraction renormalization prescription. The expression (95) is independent of the arbitrary renormalization scale κ. The renormalization group equation (85) implies that the explicit logarithmic dependence of (95) is cancelled by the κ-dependence of $g_3(\kappa)$. If κ^2 differs by orders of magnitude from $2mg\rho$, there is a large cancellation between the logarithm in (95) and the term containing $g_3(\kappa)$. Such a large cancellation can be avoided by choosing the renormalization scale to be $\kappa = \sqrt{2mg\rho}$. This is the momentum scale at which the dispersion relation for the Bogoliubov mode changes from linear to quadratic. Our final result for the energy density to second order in the quantum corrections is obtained by adding the corrections (91) and (95) to the mean-field contribution:

$$\begin{aligned}\mathcal{E}(\rho) = &\frac{1}{4}g\rho^2 + \frac{1}{60\pi^2}\frac{(2mg)^{5/2}}{m}\rho^{5/2} \\ &+\frac{1}{36}\left[g_3(\kappa) + \frac{3(4\pi - 3\sqrt{3})}{64\pi^3}\right. \\ &\left.\times\left(\ln\frac{2mg\rho}{\kappa^2} + 0.80\right)m^3g^4\right]\rho^3,\end{aligned} \tag{96}$$

It would be very difficult to measure the coupling constant $g_3(\kappa)$ experimentally by studying the 3-body scattering of atoms. It would also be difficult to calculate this parameter theoretically from a microscopic description of the interaction between atoms. Thus $g_3(\kappa)$ must be treated as a phenomenological parameter. The predictive power of the result (96) for the energy density resides in the fact that the same coupling constant $g_3(\kappa)$ appears in the second order quantum corrections to other physical quantities, such as the dispersion relation for the Bogoliubov modes. All of the low-energy observables of the Bose gas can be calculated to second order in the quantum corrections in terms of two phenomenological parameters: the S-wave scattering length a and the coupling constant $g_3(\kappa)$ associated with $3 \to 3$ scattering.

5.3 Alkali atoms

Our final result (96) for the energy density is of limited utility because it depends on the phenomenological parameter $g_3(\kappa)$. It may be possible to neglect the dependence on g_3 if the second quantum correction is dominated by the logarithm. We argue that this is indeed the case for typical alkali atoms, and that the dependence on g_3 can be eliminated in favor of a logarithmic dependence on the length scale set by the van der Waals interaction.

The interaction between two atoms at low energy can be described by a potential $V(r)$ that has a repulsive core at very short distances, an attractive region at short distances comparable to the size of the atom, and a long-range behavior given by the van der Waals potential:

$$V(R) \to -\frac{\alpha}{R^6} \cdot \tag{97}$$

The scattering length a is extremely sensitive to the short-distance behavior of the potential. Small variations in the depth or range of the potential can easily cause a to vary from $-\infty$ to $+\infty$. However, given a random distribution in one of the short-distance parameters of the potential, the distribution of a is concentrated in the region where $|a|$ is less than or comparable to the van der Waals length defined by [11]

$$\ell_{\mathrm{V}} = \left(\frac{m\alpha}{9.58}\right)^{1/4} . \tag{98}$$

For most values of the short-distance parameters, a is comparable in magnitude to ℓ_{V}. However, a varies dramatically with the short-distance parameters near the critical values at which a new 2-body bound state appears. As a parameter passes through its critical value, a approaches $\pm\infty$, changes discontinuously to $\mp\infty$, and then decreases in magnitude. The magnitude of a will be orders of magnitude larger than ℓ_{V} only if the short-distance parameter is in a narrow range around its critical value. We summarize this situation by saying that the *natural magnitude* of a is ℓ_{V}, and that much larger values require *fine-tuning* of the potential.

One can use the concept of natural magnitudes to estimate the magnitude of the coupling constants in the lagrangian for an effective field theory. A coupling constant with dimensions L^n/m, where L refers to length, will by dimensional analysis have the form $f\ell_{\mathrm{V}}^n/m$, where f is dimensionless. The assumption of naturalness is that the coefficient f is of order 1 except when a short-distance parameter of the potential is tuned to within a narrow range of a critical value. Since the coupling constants g and g_3 in the action (10) have dimensions L/m and L^4/m, respectively, the natural estimates for their magnitudes are $|g| \sim \ell_{\mathrm{V}}/m$ and $|g_3| \sim \ell_{\mathrm{V}}^4/m$, respectively.

One can improve on these naive estimates by taking into account geometrical factors of 4π. For example, using the relation (4) between the coupling constant and the scattering length a, we obtain the estimate

$$|g|_{\mathrm{natural}} \sim \frac{8\pi\ell_{\mathrm{V}}}{m} . \tag{99}$$

We argue that the magnitude of g should be comparable to this natural estimate unless the 2-body potential is tuned so that there is a bound state near threshold.

We next consider the coupling constant $g_3(\kappa)$ associated with $3 \rightarrow 3$ scattering. This is a running coupling constant that depends on an arbitrary renormalization scale κ. An estimate of the magnitude of this coupling constant must include a specification of the scale κ at which the estimate applies. Our estimates of natural values, which involve dimensional analysis, are based on the assumption that ℓ_{V} is the only important length scale. Thus the estimate must apply for momentum scales κ that correspond to the length scale ℓ_{V}. We will therefore assume that the estimate of the natural value applies to $g_3(\kappa)$ for κ comparable to $2\pi/\ell_{\mathrm{V}}$. Making a guess for the appropriate values of 4π, our estimate for the natural value is

$$|g_3(\kappa)|_{\mathrm{natural}} \sim \frac{(4\pi\ell_{\mathrm{V}})^4}{m} \quad \text{for } \kappa \sim \frac{2\pi}{\ell_{\mathrm{V}}} . \tag{100}$$

We will verify that this guess passes a simple consistency check. If both g and g_3 have natural values, then we would expect their values not to change dramatically under changes of the renormalization scale by a factor of 2 or 3. Using the solution (86) to the renormalization group equation for $g_3(\kappa)$, we see that the change in g_3 from the evolution of κ by a factor of e is

$$\Delta g_3 \sim \frac{3(4\pi - 3\sqrt{3})}{32\pi^3} m^3 g^4 . \tag{101}$$

If g has the natural magnitude given in (99), the estimate (100) is approximately equal to (101), which indicates that our guess of the factors of 4π in (100) is at least reasonable.

We now consider an atom for which g_3 has the natural magnitude given by (100). It could be much larger if the potentials describing 2-body and 3-body interactions are tuned so that there is a 3-body bound state near threshold. We cannot exclude such a possibility, but if we pick an alkali atom at random it is unlikely. We proceed to consider the three cases where g is much smaller than, comparable to, and much greater than the natural estimate given in (99). If g is much smaller than $|g|_{\mathrm{natural}}$, then renormalization has little effect on the value of g_3. The m^3g^4 term in the second order correction to the energy density in (96) is negligible compared to the g_3 term. In this case, we cannot calculate the second order correction without knowing the value of g_3. Next we consider the case where g is comparable to $|g|_{\mathrm{natural}}$. The g_3 term in (96) is then comparable in magnitude to the constant term multiplying m^3g^4. These two terms can be neglected only if the coherence length $(2mg\rho)^{-1/2}$ is orders of magnitude larger than $\ell_{\mathrm{V}}/2\pi$. In this case, the logarithmic term dominates and we obtain an estimate of the second order quantum correction that is independent of the unknown constant g_3:

$$\mathcal{E}(\rho) \approx \frac{1}{4}g\rho^2 + \frac{1}{60\pi^2}\frac{(2mg)^{5/2}}{m}\rho^{5/2} + \frac{4\pi - 3\sqrt{3}}{768\pi^3}\left(\ln\frac{mg\rho\ell_{\mathrm{V}}^2}{2\pi^2}\right) m^3 g^4 \rho^3 . \tag{102}$$

Finally, we consider the case where g is much larger than $|g|_{\mathrm{natural}}$. In this case, g_3 will quickly evolve under renormalization to a value comparable in magnitude to (101). Again we find that the g_3 term in (96) is comparable in importance to the constant under the logarithm. If the coherence length is orders of magnitude larger than the van der Waals length, these terms are negligible compared to the logarithmic term and the expression (96) for the free energy density reduces again to (102). In summary, the approximation in (102) is valid provided that g_3 is not unnaturally large compared to the estimate (100), that a is not unnaturally small compared to ℓ_{V}, and also that

$$\left|\ln(\rho a \ell_{\mathrm{V}}^2)\right| \gg 1 . \tag{103}$$

Under these conditions, we can eliminate the dependence on g_3 in favor of a logarithmic dependence on ℓ_{V}. Using (4) to express (102) in terms of the scattering length a, we obtain the expression for the energy density given in (2).

6 Conclusion

We have calculated the second order quantum correction to the energy density of a homogeneous Bose gas. This is the first correction that depends on an atomic physics parameter other than the S-wave scattering length a. We identify that parameter as a coupling constant g_3 that specifies the point-like contribution to the $3 \to 3$ scattering of atoms in the vacuum. The result for the energy density in terms of g and the running coupling constant $g_3(\kappa)$ is given in (96). In the case of alkali atoms, we argued that the dependence on g_3 can be eliminated in terms of a logarithmic dependence on the length scale ℓ_{V} set by the van der Waals interaction. The resulting expression for the energy density is given in (2).

Thus far the only alkali atoms for which Bose-Einstein condensation has been successfully carried out are ^{87}Rb, ^{23}Na, and ^{7}Li. Our calculations apply only to atoms with positive scattering length, such as ^{87}Rb and ^{23}Na. The scattering lengths for these atoms are 60 ± 15 Å for ^{87}Rb [12] and 29 ± 3 Å for ^{23}Na [13]. The parameter α in the van der Waals potential (97) is roughly 7 keVÅ^{6} for Rb and 1.8 keVÅ^{6} for Na. The van der Waals length ℓ_{V} defined in (98) is therefore about 60 Å and 30 Å, respectively. In both cases, the scattering length is comparable in magnitude to ℓ_{V}, so that one of the conditions for (102) is satisfied. The condition (103) depends on the density ρ and will be satisfied if ρ is orders of magnitude larger than $1/(a\ell_{\mathrm{V}}^2)$.

The magnitude of the quantum corrections increases with the number density. To see roughly how important these corrections are in existing magnetic traps, we evaluate them for the typical densities at the centers of the traps in the earliest experiments [1,3]. For ^{87}Rb atoms with number density $\rho = 3 \times 10^{12}/\mathrm{cm}^3$, the correction factor in (2) is $1 + 0.004 - 0.0002$. For ^{23}Na with $\rho = 3 \times 10^{14}/\mathrm{cm}^3$, the quantum correction factor in the energy density is $1 + 0.01 - 0.002$. In both cases, the second order quantum correction is an order of magnitude larger than one would have guessed by squaring the first-order quantum correction. The second order correction is relatively large because the logarithm in (2) is large, having the value -14 for ^{87}Rb and -12 for ^{23}Na. Thus the condition (103) for the validity of (2) is indeed satisfied.

The estimates given above suggest that the number densities in the Bose-Einstein condensates that have been produced thus far are not sufficiently high for the effects of quantum fluctuations on the energy density to be measurable. Since the first quantum correction scales like $\sqrt{\rho}$, the quantum corrections can be made larger by increasing the number of atoms in the trap. Unfortunately, the peak density ρ scales like $N^{2/5}$ [4], so N must be increased by orders of magnitude before the effects of quantum fluctuations on the energy density will be measurable. There are however other observables that may be more sensitive to the effects of quantum fluctuations. These effects may also be more important at temperatures near the phase transition for Bose-Einstein condensation. We hope that our explicit calculation of second-order quantum corrections for the energy density of a homogeneous Bose gas will stimulate further work on quantifying the effects of quantum fluctuations on experimentally measurable observables.

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Appendix A: Loop integrals for the energy density

In this appendix, we evaluate the integrals that are needed to calculate the second order quantum corrections to the energy density. We also list the expressions for each of the two-loop diagrams.

A.1 Energy integrals

The energy integrals can be evaluated by contour integration. The energy integral for the one-loop subdiagrams in Figures 2a–2c are

$$\int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{\omega^2 - \varepsilon^2(p) + \mathrm{i}\epsilon} = -\frac{\mathrm{i}}{2\varepsilon(\mathrm{p})}, \qquad \text{(A.1)}$$

where $\varepsilon(p)$ is the Bogoliubov dispersion relation given in (37). The energy integrals for the two-loop diagrams in Figures 2d–2g are

see equations (A.2, A.3) below.

A.2 One-loop momentum integrals

The one-loop and two-loop corrections to the ground state energy density involve momentum integrals of the form

$$I_{m,n} = \int_{\mathbf{p}} \frac{p^{2m-n}}{(p^2 + \Lambda^2)^{n/2}}, \qquad \text{(A.4)}$$

where we have introduced the following notation for

$$\int \frac{\mathrm{d}\omega_1}{2\pi} \int \frac{\mathrm{d}\omega_2}{2\pi} \frac{1}{[\omega_1^2 - \varepsilon^2(p) + \mathrm{i}\epsilon][\omega_2^2 - \varepsilon^2(q) + \mathrm{i}\epsilon][(\omega_1 + \omega_2)^2 - \varepsilon^2(r) + \mathrm{i}\epsilon]} = \frac{1}{4\varepsilon(p)\varepsilon(q)\varepsilon(r)\left[\varepsilon(p) + \varepsilon(q) + \varepsilon(r)\right]}, \qquad \text{(A.2)}$$

$$\int \frac{\mathrm{d}\omega_1}{2\pi} \int \frac{\mathrm{d}\omega_2}{2\pi} \frac{\omega_1\omega_2}{[\omega_1^2 - \varepsilon^2(p) + \mathrm{i}\epsilon][\omega_2^2 - \varepsilon^2(q) + \mathrm{i}\epsilon][(\omega_1 + \omega_2)^2 - \varepsilon^2(r) + \mathrm{i}\epsilon]} = \frac{1}{4\varepsilon(r)\left[\varepsilon(p) + \varepsilon(q) + \varepsilon(r)\right]} \cdot \qquad \text{(A.3)}$$

the integration over a momentum in D spatial dimensions:

$$\int_{\mathbf{p}} \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} \,. \tag{A.5}$$

The integrals (A.4) satisfy the identities

$$\frac{\mathrm{d}}{\mathrm{d}\Lambda^2} I_{m,n} = -\frac{n}{2} I_{m+1,n+2}, \tag{A.6}$$

$$\Lambda^2 I_{m,n} = I_{m-1,n-2} - I_{m+1,n}. \tag{A.7}$$

After integrating over angles, the integral is

$$I_{m,n} = \frac{1}{(4\pi)^{D/2}\Gamma(\frac{D}{2})} \int_0^\infty \mathrm{d}p^2 \frac{(p^2)^{m+(D-n-2)/2}}{(p^2+\Lambda^2)^{n/2}} \,. \tag{A.8}$$

If the integral is convergent, we can use integration by parts to derive the identity

$$(D+2m-n) I_{m,n} = n I_{m+2,n+2}. \tag{A.9}$$

The integral (A.8) is ultraviolet divergent if $D \geq 2(n-m)$ and infrared divergent if $D \leq n-2m$. If the integral converges, its value is

$$I_{m,n} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\frac{2n-2m-D}{2})\Gamma(\frac{D+2m-n}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{D}{2})} \Lambda^{D+2(m-n)}. \tag{A.10}$$

If the integral (A.8) is ultraviolet or infrared divergent, it can be regularized using dimensional regularization. The regularized integral is obtained by analytically continuing the expression (A.10) to $D=3$. The result is

$$I_{m,n} = \frac{1}{4\pi^2} \frac{\Gamma(\frac{2n-2m-3}{2})\Gamma(\frac{3+2m-n}{2})}{\Gamma(\frac{n}{2})} \Lambda^{3+2(m-n)}. \tag{A.11}$$

The one-loop integrals that appear in the ground state energy density are $I_{0,-1}$, $I_{-1,-1}$, and $I_{1,1}$. In 3 dimensions, these integrals have power ultraviolet divergences. With dimensional regularization, they are given by the finite expressions

$$I_{0,-1} = \frac{1}{15\pi^2}\Lambda^5, \tag{A.12}$$

$$I_{-1,-1} = -\frac{1}{6\pi^2}\Lambda^3, \tag{A.13}$$

$$I_{1,1} = \frac{1}{3\pi^2}\Lambda^3. \tag{A.14}$$

A.3 Two-loop momentum integrals

The two-loop correction to the ground state energy density (51) involves a linear combination of the integrals

$$J_{l,m,n} = \int_{\mathbf{p}}\int_{\mathbf{q}} \frac{\left(p/\sqrt{p^2+\Lambda^2}\right)^l \left(q/\sqrt{q^2+\Lambda^2}\right)^m \left(r/\sqrt{r^2+\Lambda^2}\right)^n}{p\sqrt{p^2+\Lambda^2}+q\sqrt{q^2+\Lambda^2}+r\sqrt{r^2+\Lambda^2}}, \tag{A.15}$$

where $r = |\mathbf{p}+\mathbf{q}|$. In $D=3$, these integrals have quartic and quadratic ultraviolet divergences that cancel in the combination of integrals J given in (45). The expression for J can be written

$$\begin{aligned} J = \int_{\mathbf{p}}\int_{\mathbf{q}} & \frac{1}{p\sqrt{p^2+\Lambda^2}+q\sqrt{q^2+\Lambda^2}+r\sqrt{r^2+\Lambda^2}} \\ & \times \left[\frac{6p}{\sqrt{p^2+\Lambda^2}} - \frac{2\sqrt{p^2+\Lambda^2}}{p} \right. \\ & - \frac{3pqr}{\sqrt{p^2+\Lambda^2}\sqrt{q^2+\Lambda^2}\sqrt{r^2+\Lambda^2}} \\ & \left. - \frac{p\sqrt{q^2+\Lambda^2}\sqrt{r^2+\Lambda^2}}{qr\sqrt{p^2+\Lambda^2}} \right]. \end{aligned} \tag{A.16}$$

This integral still has linear and logarithmic ultraviolet divergences. By subtracting and adding appropriate terms in the integrand of J, we can isolate the linear and logarithmic divergences into separate terms:

$$J = J_{\text{lin}} + J_{\text{log}} + J_{\text{num}}. \tag{A.17}$$

The term containing the linear ultraviolet divergence is

$$J_{\text{lin}} = 2\int_{\mathbf{p}} \left[2 - \frac{p}{\sqrt{p^2+\Lambda^2}} - \frac{\sqrt{p^2+\Lambda^2}}{p} \right] \int_{\mathbf{q}} \frac{1}{q^2} \,. \tag{A.18}$$

The term in (A.17) containing the logarithmic ultraviolet divergence is

$$\begin{aligned} J_{\text{log}} = & -\Lambda^4 \int_{\mathbf{p}}\int_{\mathbf{q}} \left\{ \frac{2}{(p^2+\Lambda^2)(q^2+\Lambda^2)(p^2+q^2+k^2+2\Lambda^2)} \right. \\ & \left. + \left[\frac{1}{p^2+q^2+k^2+2\Lambda^2} - \frac{1}{2(q^2+\Lambda^2)} \right] \frac{1}{(p^2+\Lambda^2)^2} \right\}. \end{aligned} \tag{A.19}$$

The integral J_{num} obtained by subtracting (A.18) and (A.19) from (A.16) is convergent in $D=3$ dimensions and can be evaluated numerically. It is convenient to symmetrize the integrand over the six permutations of p, q, and r in order to avoid cancellations between different

$$\begin{aligned} J_{\mathrm{num}} = \int_{\mathbf{p}}\int_{\mathbf{q}} \frac{1}{6}\sum_{(pqr)} \Bigg\{ & \frac{1}{p\sqrt{p^2+\Lambda^2}+q\sqrt{q^2+\Lambda^2}+r\sqrt{r^2+\Lambda^2}} \left[\frac{6p}{\sqrt{p^2+\Lambda^2}} - \frac{2\sqrt{p^2+\Lambda^2}}{p} \right. \\ & \left. - \frac{3pqr}{\sqrt{p^2+\Lambda^2}\sqrt{q^2+\Lambda^2}\sqrt{r^2+\Lambda^2}} - \frac{p\sqrt{q^2+\Lambda^2}\sqrt{r^2+\Lambda^2}}{qr\sqrt{p^2+\Lambda^2}} \right] + \frac{2}{q^2}\left[2 - \frac{p}{\sqrt{p^2+\Lambda^2}} - \frac{\sqrt{p^2+\Lambda^2}}{p} \right] \\ & + \frac{2\Lambda^4}{(p^2+q^2+r^2+2\Lambda^2)(p^2+\Lambda^2)(q^2+\Lambda^2)} + \left[\frac{1}{p^2+q^2+r^2+2\Lambda^2} - \frac{1}{2(q^2+\Lambda^2)} \right] \frac{\Lambda^4}{(p^2+\Lambda^2)^2} \Bigg\}. \end{aligned} \tag{A.20}$$

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regions of momentum space. The resulting expression is

see equation (A.20) above.

Since Λ is the only scale in the integrand, dimensional analysis implies that the integral is proportional to Λ^4. Evaluating the coefficient of Λ^4 numerically, we obtain

$$J_{\mathrm{num}} = 2.10 \times 10^{-3}\Lambda^4. \tag{A.21}$$

Because of the severe cancellations between the various terms in the integral, we were only able to calculate it to 3 significant figures.

We evaluate the ultraviolet divergent integrals J_{lin} in (A.18) and $J_{\log}$ in (A.19) using dimensional regularization. The integral over $\mathbf{q}$ in (A.18) vanishes since there is no scale in the integrand, and therefore $J_{\mathrm{lin}} = 0$. The integral (A.19) is evaluated in Appendix A.4 in the limit $D \to 3$, and the result is

$$J_{\log} = \frac{4\pi - 3\sqrt{3}}{192\pi^3}\left(\frac{1}{D-3} - 1.13459\right)\Lambda^{4+2(D-3)}. \tag{A.22}$$

Adding (A.21) and (A.22), we obtain the complete result for J using dimensional regularization:

$$J = \frac{4\pi - 3\sqrt{3}}{192\pi^3}\left(\frac{1}{D-3} + 0.57\right)\Lambda^{4+2(D-3)}. \tag{A.23}$$

A.4 Evaluation of $J_{\log}$

The term (A.19), which contains the logarithmic ultraviolet divergence in the integral J, can be written

$$J_{\log} = -\Lambda^4(2K_1 + K_2), \tag{A.24}$$

where K_1 and K_2 are the following integrals:

$$K_1 = \int_{\mathbf{p}}\int_{\mathbf{q}} \frac{1}{(p^2+q^2+r^2+2\Lambda^2)(p^2+\Lambda^2)(q^2+\Lambda^2)}, \tag{A.25}$$

$$K_2 = \int_{\mathbf{p}}\int_{\mathbf{q}} \left[\frac{1}{p^2+q^2+r^2+2\Lambda^2} - \frac{1}{2(q^2+\Lambda^2)}\right]\frac{1}{(p^2+\Lambda^2)^2}, \tag{A.26}$$

where $r = |\mathbf{p}+\mathbf{q}|$.

We first consider the integral K_1. Setting $r^2 = p^2 + q^2 + 2\mathbf{p}\cdot\mathbf{q}$ and then introducing Feynman parameters, the integral (A.25) becomes

$$\begin{aligned} K_1 = & \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \\ & \times \int_{\mathbf{p}}\int_{\mathbf{q}} \frac{1}{[(1-y)p^2 + (1-x)q^2 + z\mathbf{p}\cdot\mathbf{q} + \Lambda^2]^3}, \end{aligned} \tag{A.27}$$

where $z = 1 - x - y$. Dimensional regularization allows us to shift and rescale the momentum variables. We can eliminate the dot product from the denominator by making the shift $\mathbf{p} \to \mathbf{p} - \frac{z}{2(1-y)}\mathbf{q}$. After rescaling $\mathbf{p}$ by $(1-y)^{-1/2}$ and $\mathbf{q}$ by $\left(\frac{(1-x)(1-y)-z^2/4}{1-y}\right)^{-1/2}$, the integral factors into a Feynman parameter integral and an integral over the momenta:

$$\begin{aligned} K_1 = & \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y[(1-x)(1-y) - z^2/4]^{-D/2} \\ & \times \int_{\mathbf{p}}\int_{\mathbf{q}} \frac{1}{(p^2+q^2+\Lambda^2)^3}\,. \end{aligned} \tag{A.28}$$

The integral over the momenta can be evaluated analytically:

$$\int_{\mathbf{p}}\int_{\mathbf{q}} \frac{1}{(p^2+q^2+\Lambda^2)^3} = \frac{\Gamma(3-D)}{2(4\pi)^D}\Lambda^{2(D-3)}. \tag{A.29}$$

The gamma function has a pole at $D = 3$. To obtain K_1 in the limit $D \to 3$, we need to expand the Feynman parameter integral in powers of $D - 3$:

$$\begin{aligned} K_1 = & \frac{\Gamma(3-D)}{2(4\pi)^D}\Lambda^{2(D-3)} \\ & \times \left\{ \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \left[(1-x)(1-y) - \frac{z^2}{4}\right]^{-3/2} \right. \\ & - \frac{D-3}{2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \left[(1-x)(1-y) - \frac{z^2}{4}\right]^{-3/2} \\ & \left. \times \ln\left[(1-x)(1-y) - \frac{z^2}{4}\right] \right\}, \end{aligned} \tag{A.30}$$

where $z = 1 - x - y$. The first integral in (A.30) can be computed analytically and has the value $4\pi/3$. The second integral has to be computed numerically and has the value -9.43698. Extracting the pole in $D-3$ from the gamma function in (A.30) and keeping all terms that survive in the limit $D \to 3$, we obtain

$$K_1 = -\frac{1}{96\pi^2} \Lambda^{2(D-3)} \left[\frac{1}{D-3} + 1.12646 + \gamma - \ln(4\pi) \right], \tag{A.31}$$

where γ is Euler's constant.

We next consider the integral K_2 in (A.26). By introducing a Feynman parameter, it can be written

$$K_2 = \int_0^1 \mathrm{d}x (1-x) \int_{\mathbf{p}} \int_{\mathbf{q}} \left(\frac{1}{[p^2 + xq^2 + x\mathbf{p}\cdot\mathbf{q} + \Lambda^2]^3} - \frac{1}{[(1-x)p^2 + xq^2 + \Lambda^2]^3} \right). \tag{A.32}$$

By shifting and rescaling the momentum variables, we can reduce the integral over the momenta to (A.29). In the first term of (A.32), we shift $\mathbf{p} \to \mathbf{p} - \frac{x}{2}\mathbf{q}$ and then rescale $\mathbf{q}$ by $\left(\frac{x(4-x)}{4}\right)^{-1/2}$. In the second term, we rescale $\mathbf{p}$ by $(1-x)^{-1/2}$ and $\mathbf{q}$ by $x^{-1/2}$. After integrating over $\mathbf{p}$ and $\mathbf{q}$, we obtain

$$K_2 = \frac{\Gamma(3-D)}{2(4\pi)^D} \Lambda^{2(D-3)} \times \int_0^1 \mathrm{d}x (1-x) x^{-D/2} \left[(1-x/4)^{-D/2} - (1-x)^{-D/2} \right]. \tag{A.33}$$

To obtain K_2 in the limit $D \to 3$, we need to expand the integrand in (A.33) in powers of $D-3$:

$$K_2 = \frac{\Gamma(3-D)}{2(4\pi)^D} \Lambda^{2(D-3)} \left\{ \int_0^1 \mathrm{d}x \left(\frac{8(1-x)}{\sqrt{x^3(4-x)^3}} - \frac{1}{\sqrt{x^3(1-x)}} \right) - \frac{D-3}{2} \int_0^1 \mathrm{d}x \left(\frac{8(1-x)}{\sqrt{x^3(4-x)^3}} \ln \frac{x(4-x)}{4} - \frac{1}{\sqrt{x^3(1-x)}} \ln[x(1-x)] \right) \right\}. \tag{A.34}$$

The integrals can be evaluated analytically. Extracting the pole in $D-3$ from the gamma function and keeping all terms that survive in the limit $D \to 3$, we obtain

$$K_2 = \frac{\sqrt{3}}{64\pi^3} \Lambda^{2(D-3)} \times \left[\frac{1}{D-3} + \frac{4\pi}{3\sqrt{3}} - \frac{1}{2} \ln \frac{3}{4} - 1 + \gamma - \ln(4\pi) \right]. \tag{A.35}$$

A.5 Two-loop diagrams for the free energy density

The two-loop vacuum diagrams that contribute to $-\mathrm{i}\Omega_2(\mu, \overline{v}_0)$ are shown in Figure 2. These diagrams can be reduced to momentum integrals by integrating over the loop energies using the identities in Appendix A.1. Expressed in terms of the integrals $I_{m,n}$ and $J_{l,m,n}$ defined in (A.4) and (A.15), the contributions of the individual diagrams to $\Omega_2(\mu, \overline{v}_0)$ are

$$\Omega_2^{(a)} = \frac{3}{64} g I_{1,1}^2 \tag{A.36}$$

$$\Omega_2^{(b)} = \frac{1}{32} g I_{-1,-1} I_{1,1} \tag{A.37}$$

$$\Omega_2^{(c)} = \frac{3}{64} g I_{-1,-1}^2 \tag{A.38}$$

$$\Omega_2^{(d)} = -\frac{3}{16} m g^2 \overline{v}_0^2 J_{1,1,1} \tag{A.39}$$

$$\Omega_2^{(e)} = \frac{3}{8} m g^2 \overline{v}_0^2 J_{0,0,1} \tag{A.40}$$

$$\Omega_2^{(f)} = -\frac{1}{16} m g^2 \overline{v}_0^2 J_{-1,-1,1} \tag{A.41}$$

$$\Omega_2^{(g)} = -\frac{1}{8} m g^2 \overline{v}_0^2 J_{-1,0,0}. \tag{A.42}$$

Adding up these diagrams, we get (44).

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